Finitely presented quadratic algebras of intermediate growth

Dilber Koçak

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Abstract

In this article, we give two examples of finitely presented quadratic algebras (algebras presented by quadratic relations) of intermediate growth.

1   Introduction

Let A be a finitely generated algebra over a field k with generating set $S = \{x_1, \ldots, x_m\}$. We denote by A_n the subspace of elements of degree at most n, then $A = \bigcup_{n=0}^{\infty} A_n$. The growth function γ_A^S of A with respect to S is defined as the dimension of the vector space A_n over k,

$$\gamma_A^S(n) = dim_k(A_n)$$

The function γ_A^S depends on the generating set S. This dependence can be removed by introducing an equivalence relation: Let f and g be eventually monotone increasing and positive valued functions on $\mathbb{N}$. Set $f \preceq g$ if and only if there exist $N > 0$, $C > 0$, such that $f(n) \leq g(Cn)$, for $n \geq N$, and $f \sim g$ if and only if $f \preceq g$ and $g \preceq f$. The equivalence class of f is called the *growth rate* of f. Simple verification shows that growth functions of an algebra with respect to different generating sets are equivalent.

The growth rate is a useful invariant for finitely generated algebraic structures such as groups, semigroups and algebras. The notion of growth function for groups was introduced by Schwarz [Šva55] and independently by Milnor [Mil68]. The description of groups of polynomial growth was obtained by Gromov in his celebrated work [Gro81]. He proved that every finitely generated group of polynomial growth contains a nilpotent subgroup of finite index.

The study of growth of algebras dates back to the papers by Gelfand and Kirillov, [GK66a, GK66b]. In this paper we are mainly interested in finitely presented algebras whose growth functions behave in intermediate way i.e., they grow faster than any polynomial function but slower than any exponential function. Govorov gave the first examples of finitely generated semigroups and associative algebras of intermediate growth in [Gov72]. Examples of algebras of intermediate growth can also be found in [Ste75, Smi76, She80, Ufn80, KKM83]. The first examples of finitely generated groups of intermediate growth were constructed by Grigorchuk [Gri83, Gri84]. It is still an open problem whether there exists a finitely presented group of intermediate growth. In contrast, there are examples of finitely presented algebras of intermediate growth. The first example is the universal enveloping algebra of a Lie algebra W with basis $\{w_{-1}, w_0, w_1, w_2, \ldots\}$ and brackets defined by $[w_i, w_j] = (i - j)w_{i+j}$. W is a subalgebra of the generalized Witt algebra $W_{\mathbb{Z}}$ (see [AS74, p.206] for definitions). It was proven in [Ste75] that W has a finite presentation with two generators and six relations. It is also a graded algebra with generators of degree -1 and 2. Since W has linear growth, its universal enveloping algebra is an example of finitely presented associative algebra of intermediate growth.

The main goal of this paper is to present examples of finitely presented quadratic algebras (algebras defined by quadratic relations) of intermediate growth. The class of quadratic algebras contains a class

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of finitely presented algebras, called *Koszul algebras*. They play an important role in many studies. In [PP05], it is conjectured that the Hilbert series of a Koszul algebra A is a rational function and in particular, the growth of A is either polynomial or exponential.

In order to construct our first example of a finitely presented quadratic algebra of intermediate growth, we consider the Kac-Moody algebra for the generalized Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. This is a graded Lie algebra of polynomial growth whose generators are of degree 1. Next, we consider a suitable subalgebra and its universal enveloping algebra.

Theorem 1 *Let U be the associative algebra with generators x, y and relations $x^3y - 3x^2yx + 3xyx^2 - yx^3 = 0$, $y^3x - 3y^2xy + 3yxy^2 - xy^3 = 0$. Then*

(i) *It is the universal enveloping algebra of a subalgebra of the the Kac-Moody algebra for the generalized Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.*

(ii) *U is a graded algebra with generators of degree 1.*

(iii) *It has intermediate growth of type $e^{\sqrt{n}}$.*

(iv) *The Veronese subalgebra $V_4(U)$ of U is a quadratic algebra given by 14 generators and 96 quadratic relations and it has the same growth type with U.*

The Kac-Moody algebra for the generalized Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ is the affine Lie algebra $A_1^{(1)}$. (For the definition of Kac-Moody algebras and classification of affine Lie algebras see [Kac85]). It has a subalgebra which is isomorphic to the Lie subalgebra L of $sl_2(\mathbb{C}[t])$ which consists of all matrices with entries on and under the diagonal divisible by t. That is,

$$L = \{a = (a_{ij})_{2 \times 2} \mid a_{ij} \in \mathbb{C}[t], \ tr(a) = 0 \text{ and for } (i,j) \neq (1,2), \ t \text{ divides } a_{ij}\}$$

with the usual Lie bracket $[a, b] = ab - ba$. It follows from [Kac85, Theorem 9.11] that L is finitely presented.

In this paper we will prove this by using the axioms of Lie bracket without mentioning the theory of Kac-Moody algebras. In Section 2 we show that L is a finitely presented graded Lie algebra whose generators are all of degree 1 and L has linear growth. In Section 3 we explain the relation between the growth of a Lie algebra and its universal enveloping algebra. In Section 4 we consider the Veronese subalgebra of U to obtain a finitely presented quadratic algebra of intermediate growth and in Section 5 we complete the proof of Theorem 1. In Section 6 we give another example of finitely presented associative algebra A of intermediate growth related to the example of the monoid in [Kob95]. A has the following presentation:

$$A = \langle a, b, c \mid b^2a = ab^2, \ b^2c = aca, \ acc = 0, \ aba = 0, \ abc = 0, \ cba = 0, \ cbc = 0 \rangle$$

We show that A has intermediate growth of type $e^{\sqrt{n}}$ and its Veronese subalgebra $V_3(A)$ is an example of finitely presented quadratic algebra of intermediate growth. In Section 7 , we give an explicit presentation of the Veronese subalgebra $V_4(U)$ of the first construction U as an example of a finitely presented quadratic algebra of intermediate growth.

2   An example of a finitely presented Lie Algebra of linear growth

The following example is a subalgebra of the Kac-Moody Algebra for the generalized Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ [Kac85].

Consider the subalgebra L of $Sl_2(\mathbb{C}[t])$ over $\mathbb{C}$ (i.e., matrices of trace 0 with entries in $\mathbb{C}[t]$)) which consists of matrices whose entries on and under the diagonal are divisible by t. That is,

$$L = \{a = (a_{ij})_{2x2} \mid a_{ij} \in \mathbb{C}[t], \ tr(a) = 0 \text{ and for } (i,j) \neq (1,2), \ t \text{ divides } a_{ij}\}$$

with the usual Lie bracket $[a, b] = ab - ba$.

Proposition 1 *Let L be the Lie algebra described above. Then it has the following properties.*

(i) *L is finitely presented with generators* $x := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $y := \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$ *and the defining relations* $[x, [x, [x, y]]] = 0$ *and* $[y, [y, [y, x]]] = 0$.

(ii) *$L = \bigoplus_{k \geq 1} L_k$ is graded and generated by L_1.*

(iii) *L has linear growth.*

Proof: Take $x_1 := x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y_1 := y = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$, and let $z_1 := \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$.

In fact, define $x_i := \begin{pmatrix} 0 & t^{i-1} \\ 0 & 0 \end{pmatrix}$, $y_i := \begin{pmatrix} 0 & 0 \\ t^i & 0 \end{pmatrix}$, and let $z_i := \begin{pmatrix} t^i & 0 \\ 0 & -t^i \end{pmatrix}$ for $i \geq 1$.

An arbitrary element $w \in L$ is of the form:

$$w = \begin{pmatrix} \sum_{i=1}^n m_i t^i & \sum_{i=1}^n k_i t^{i-1} \\ \sum_{i=1}^n l_i t^i & \sum_{i=1}^n -m_i t^i \end{pmatrix} = \sum_{i=1}^n k_i x_i + \sum_{i=1}^n l_i y_i + \sum_{i=1}^n m_i z_i$$

So, any element of L can be written as a linear combination of x_i, y_i, z_i for $i \geq 1$ and $\{x_i, \ y_i, \ z_i\}_{i=1}^\infty$ forms a linearly independent set over $\mathbb{C}$. L has the following relations

$$[x_i, y_j] = z_{i+j-1}, \tag{1}$$

$$[x_i, z_j] = -2x_{i+j}, \tag{2}$$

$$[y_i, z_j] = 2y_{i+j}, \tag{3}$$

$$[x_i, x_j] = 0, \tag{4}$$

$$[y_i, y_j] = 0, \tag{5}$$

$$[z_i, z_j] = 0. \tag{6}$$

for $i, j \geq 1$. In particular,

$$x_{i+1} = -\frac{1}{2}[x_i, z_1],$$

$$y_{i+1} = \frac{1}{2}[y_i, z_1],$$

$$z_i = [x_i, y_1].$$

It follows that L is generated by x_1 and y_1. In order to show that all the relations (1) - (6) can be derived from the relations $[x_1, [x_1, [x_1, y_1]]] = 0$ and $[y_1, [y_1, [y_1, x_1]]] = 0$, we apply induction on $i + j = n$. If $i + j = 2$, the relations (1) - (6) hold trivially. If $i + j = 3$,

3

$$
\begin{aligned}
[x_1, y_2] &= [x_1, \tfrac{[y_1, z_1]}{2}] \\
&= -\tfrac{1}{2}\left([z_1, [x_1, y_1]] + [y_1, [z_1, x_1]]\right) \\
&= [x_2, y_1] \\
&= z_2
\end{aligned}
$$

$$
\begin{aligned}
[x_1, z_2] &= [x_1, [x_2, y_1]] \\
&= -[y_1, [x_1, x_2]] + [x_2, [y_1, x_1]] \quad \text{(since } [x_1, x_2] = 0) \\
&= [x_2, [x_1, y_1]] \\
&= [x_2, z_1] \\
&= -2x_3
\end{aligned}
$$

$$
\begin{aligned}
[y_1, z_2] &= [y_1, [x_1, y_2]] \\
&= -([y_2, [y_1, x_1]] + [x_1, [y_2, y_1]]) \quad \text{(since } [y_1, y_2] = 0) \\
&= [y_2, z_1] \\
&= 2y_3
\end{aligned}
$$

The relations (4)-(5) for $n = 3$ correspond to relations of L_0. Observe the following three equations for $[z_2, z_1]$,

$$
\begin{aligned}
[z_2, z_1] &= [[x_2, y_1], z_1] \\
&= -([[z_1, x_2], y_1] + [[y_1, z_1], x_2]) \\
&= [[x_2, z_1], y_1] + [x_2, [y_1, z_1]] \\
&= -2[x_3, y_1] + 2[x_2, y_2] \\
&= k
\end{aligned}
$$

$$
\begin{aligned}
[z_2, z_1] &= [[x_1, y_2], z_1] \\
&= -([[z_1, x_1], y_2] + [[y_2, z_1], x_1]) \\
&= [[x_1, z_1], y_2] + [x_1, [y_2, z_1]] \\
&= -2[x_2, y_2] + 2[x_1, y_3] \\
&= l
\end{aligned}
$$

$$
\begin{aligned}
[z_2, z_1] &= [z_2, [x_1, y_1]] \\
&= -([y_1, [z_2, x_1]] + [x_1, [y_1, z_2]]) \\
&= 2[x_3, y_1] - 2[x_1, y_3] \\
&= m
\end{aligned}
$$

$3.[z_2, z_1] = k + l + m = 0$. So, (1) - (6) hold for $n = 3$. Now, suppose that (1) - (6) hold for $i + j \le n$ for some $n \ge 3$. For $1 \le i \le n - 1$,

$$
\begin{aligned}
[x_i, y_{j+1}] &= \tfrac{1}{2}[x_i, [y_j, z_1]] \\
&= -\tfrac{1}{2}([z_1, [x_i, y_j]] + [y_j, [z_1, x_i]]) \\
&= [x_{i+1}, y_j]
\end{aligned}
$$

$$
\begin{aligned}
-2x_{n+1} &= [x_n, z_1] \\
&= -\tfrac{1}{2}[[x_1, z_{n-1}], z_1] \\
&= \tfrac{1}{2}([[z_1, x_1], z_{n-1}] + [[z_{n-1}, z_1], x_1]) \\
&= [x_2, z_{n-1}]
\end{aligned}
$$

and,

$$
\begin{aligned}
[x_i, z_{j+1}] &= [x_i, [x_1, y_{j+1}]] \\
&= -([y_{j+1}, [x_i, x_1]] + [x_1, [y_{j+1}, x_i]]) \\
&= [x_1, z_{i+j}]
\end{aligned}
$$

Similarly, it can be shown that

4

$$2y_{n+1} = [y_i, z_{j+1}]$$

for any $i, j \geq 1$ such that $i + j = n$. So (1) - (3) hold for $i + j = n + 1$.

$$
\begin{aligned}
[x_1, x_n] &= -\tfrac{1}{2}[x_1, [x_i, z_j]] \\
&= \tfrac{1}{2}([z_j, [x_1, x_i]] + [x_i, [z_j, x_1]]) \\
&= -\tfrac{1}{2}[x_i, [x_1, z_j]] \\
&= [x_i, x_j]
\end{aligned}
$$

This equality implies $[x_i, x_j] = [x_j, x_i]$. Similarly, one checks that $[y_i, y_j] = [y_j, y_i]$. Hence, (4) - (5) hold for $i + j = n + 1$.

Finally, we need check that (6) holds for $i + j = n + 1$.

$$
\begin{aligned}
[z_1, z_n] &= [z_1, [x_n, y_1]] &&= 2[x_{n+1}, y_1] - 2[x_n, y_2] \\
&= [z_1, [x_{n-1}, y_2]] &&= 2[x_n, y_2] - 2[x_{n-1}, y_3] \\
&\vdots \\
&= [z_1, [x_1, y_n]] &&= 2[x_2, y_n] - 2[x_1, y_{n+1}]
\end{aligned}
$$

implies that $n.[z_1, z_n] = 2[x_{n+1}, y_1] - 2[x_1, y_{n+1}]$ and,

$$
\begin{aligned}
2[x_1, y_{n+1}] &= [x_1, [y_1, z_n]] &&= -[z_n, [x_1, y_1]] - [y_1, [z_n, x_1]] \\
&&&= [z_1, z_n] + 2[x_{n+1}, y_1]
\end{aligned}
$$

So $[z_1, z_n] = 0$. Now, consider $[z_i, z_j]$ for $i \in \{1, \ldots, n-1\}$,

$$
\begin{aligned}
[z_i, z_j] &= [z_i, [x_j, y_1]] &&= -([y_1, [z_i, x_j]] + [x_j, [y_1, z_i]]) \\
&&&= 2[x_{i+j}, y_1] - 2[x_j, y_{i+1}]
\end{aligned}
$$

and,

$$
\begin{aligned}
[x_j, y_{i+1}] &= \tfrac{1}{2}[x_j, [y_i, z_1]] &&= -\tfrac{1}{2}([z_1, [x_j, y_i]] + [y_i, [z_1, x_j]]) \\
&&&= -\tfrac{1}{2}([z_1, z_n] + [y_i, 2x_{j+1}]) \\
&&&= [x_{j+1}, y_i]
\end{aligned}
$$

By applying this i times we get $[x_j, y_{i+1}] = [x_n, y_1]$, so that

$$[z_i, z_j] = 0 \text{ for } i + j = n + 1$$

i.e., (6) holds for $i + j = n + 1$. By (1) - (3), the set $\{x_i, y_i, z_i\}_{i=1}^{\infty}$ forms a basis for L as a vector space. It can be observed that $L = \bigoplus_{k \geq 1} L_k$ where $L_{2k-1} = \langle x_k \rangle \oplus \langle y_k \rangle$ and $L_k = \langle z_k \rangle$ for $k \geq 1$. Since

$$[L_{2k-1}, L_{2m-1}] \subseteq L_{2(k+m-1)},$$

$$[L_{2k}, L_{2m}] = 0,$$

$$[L_{2k-1}, L_{2m}] \subseteq L_{2(k+m)-1},$$

L admits an $\mathbb{N}$-gradation given by the sum of occurrences of x and y in each commutator i.e., $L = \bigoplus_{k \geq 1} L_k$ is a graded Lie algebra generated by two elements of degree 1 ($deg(a) = min\{n | a \in \bigoplus_{k=1}^{n} L_k)\}$) and L has linear growth ($dim\, L_i \in \{1, 2\}$ for $i \geq 1$).

Remark 1 *We notice that L also admits a $\mathbb{Z}$-gradation. It is a 3-graded Lie algebra (in the sense of [dO03]) over $\mathbb{C}$ generated by elements x of degree 1 and y of degree -1.*

3 The relation between the growth of a Lie algebra and its universal enveloping algebra

Let L be any Lie algebra over a field k and $U(L)$ be its universal enveloping algebra. For an ordered basis $u_1, u_2, \ldots$ of L, monomials $u_{i_1} \ldots u_{i_r}$ with $i_1 \leq i_2 \leq \cdots \leq i_r$ form a basis for $U(L)$ (Poincaré-Birkhoff-Witt Theorem ([Ber78])). If $L = \bigoplus L_n$ is a graded Lie algebra such that all the components are finite dimensional, then

$$\sum_{n=0}^{\infty} b_n t^n = \prod_{n=1}^{\infty} (1 - t^n)^{-a_n} \tag{7}$$

where $a_n := dim(L_n)$ and $b_n :=$ number of monomials of length n in $U(L)$ ([Smi76]). The proof of the following proposition can be found in various papers ([Ber83], [Pet93], [BG00]).

Proposition 2 *If a_n and b_n are related by (7) and $a_n \sim n^d$, then $b_n \sim e^{n^{\frac{d+1}{d+2}}}$.*

Corollary 1 *If a Lie algebra L grows polynomially then its universal enveloping algebra $U(L)$ has intermediate growth. In particular, if L has linear growth, then $U(L)$ has growth of type $e^{\sqrt{n}}$.*

4 Veronese subalgebra of an associative graded algebra

Let $A = k\langle x_1, \ldots, x_m \rangle$ be a free associative algebra over a field k with generating set $\{x_1, \ldots, x_m\}$. Each element u of A can be written uniquely as

$$u = u_0 + u_1 + \cdots + u_l,$$

where $A_0 = k$, $u_i \in A_i$ and A_i is the vector space over k spanned by m^i monomials of length i. Let $R = \{f_1, f_2, \ldots, f_s\}$ be a finite set of non-zero homogeneous polynomials and I be the ideal generated by R. Since I is generated by homogeneous polynomials, the factor algebra $\tilde{A} = A/I$ is graded:

$$\tilde{A} = \tilde{A}_0 \oplus \tilde{A}_1 \oplus \cdots \oplus \tilde{A}_n \oplus \ldots$$

where $\tilde{A}_i = (A_i + I)/I \cong A_i/(A_i \cap I)$. For $d \geq 1$, a *Veronese subalgebra* of $\tilde{A}$ is defined as

$$V_d(\tilde{A}) := k \oplus \tilde{A}_d \oplus \tilde{A}_{2d} \oplus \ldots$$

It is straightforward to see that,

$$growth\ of\ \tilde{A} \sim growth\ of\ V_d(\tilde{A})$$

Proposition 3 *[BF85] For sufficiently large d, $V_d(\tilde{A})$ is quadratic.*

Proof: Let $d_1, \ldots, d_s$ be the degrees of $f_1, f_2, \ldots, f_s$ respectively and $d \geq max\{d_i,\ 1 \leq i \leq s\}$. For any two words v', v'' such that

$$deg(v') + d_i + deg(v'') = d$$

consider the element $v' f_i v'' \in A_d$, and for any two words w', w'' such that

$$deg(w') + d_i + deg(w'') = 2d$$

consider the element $w' f_i w'' \in A_{2d}$. Let $R^* = \{v' f_i v'', w' f_i w''\}$ for $i \in \{1, \ldots, s\}$ and a be a homogeneous element from $A^{(n)} \cap I$. Say $a = \sum \alpha v f_i w$, where $\alpha \in k$, v and w are words. If we choose a summand and represent $v = v_1 v_2$, $deg(v_1)$ is a multiple of d, $0 \leq deg(v_2) < d$. Similarly, $w = w_2 w_1$, $deg(w_1)$ is a multiple of d, $0 \leq deg(w_2) < d$. Then we will get $deg(v_2 f_i w_2) = d$ or $2d$. Hence $v_2 f_i w_2 \in R^*$. It shows that $V_d(A) \cap I$ is an ideal generated by the elements of R^* and an element $v' f_i v''$ is a linear combination of free generators of $A^{(n)}$ whereas $w' f_i w''$ is a quadratic element in these generators. So $V_d(\tilde{A}) = V_d(A)/(V_d(A) \cap I)$ is a quadratic algebra.

5 Proof of Theorem 1

Let $L = \langle x_1, \ldots, x_m \mid f_1 = 0, \ldots, f_r = 0 \rangle$ where each of f_i is a linear combination of the commutators (elements of the form $[x_{i_1}, \ldots, x_{i_k}]$ with an arbitrary distribution of parentheses inside). Then the universal enveloping algebra $U(L)$ of L is an associative algebra with the identical set of generators and relations, where the commutators are thought of as in the ordinary associative sense: $[x, y] = xy - yx$ [Bou89, Proposition 2, p.14]. The universal enveloping algebra $U(L)$ of $L = \langle x_1, y_1 \mid [x_1, [x_1, [x_1, y_1]]] = 0, [y_1, [y_1, [y_1, x_1]]] = 0 \rangle$ has the following presentation:

$$U(L) = \langle x_1, y_1 \mid x_1^3 y_1 - 3x_1^2 y_1 x_1 + 3x_1 y_1 x_1^2 - y_1 x_1^3 = 0,\ y_1^3 x_1 - 3y_1^2 x_1 y_1 + 3y_1 x_1 y_1^2 - x_1 y_1^3 = 0 \rangle.$$

So, the associative algebra U in Theorem 1 is the universal enveloping algebra $U(L)$ of L. By Proposition 2, since L has linear growth, the growth rate of $U(L)$ is intermediate of type $e^{\sqrt{n}}$. In order to obtain a quadratic algebra of intermediate growth we consider a Veronese subalgebra of $V_4(U)$ as explained in the previous section and conclude that for a given finitely presented graded algebra with all generators of degree 1, one can construct a finitely presented graded algebra with all relations of degree 2. $V_4(U)$ is an example of a finitely presented graded algebra with intermediate growth. It has 14 generators and 96 relations. In the next section we compute all these relations.

6 A construction based on Kobayashi's example

In this section we construct another example of a finitely presented associative algebra with quadratic relations whose growth function is intermediate. For this, we consider the following example of a monoid with 0 that appears in the paper of Kobayashi [Kob95].

$$M = \langle a, b, c \mid ba = ab,\ bc = aca,\ acc = 0 \rangle$$

where $w(a) = w(c) = 1$, $w(b) = 2$, w is a positive weight function on M. Kobayashi shows that M is a finitely presented monoid with solvable word problem which cannot be presented by a regular complete system. In order to prove that it cannot be presented by a regular complete system, he proves that M has intermediate growth. Now, we consider the semigroup algebra $k[M]$ over a field k. $k[M]$ has the same presentation and growth function with M. So $k[M]$ is an example of finitely presented associative graded algebra of intermediate growth. But the generators of $k[M]$ have degrees $deg(a) = deg(c) = 1$ and $deg(b) = 2$. To construct a quadratic algebra with these properties, we need to consider an algebra whose generators are all of degree 1. Thus we consider the following monoid:

$$\tilde{M} = \langle a, b, c \mid b^2 a = ab^2,\ b^2 c = aca,\ acc = 0,\ aba = 0,\ abc = 0,\ cba = 0,\ cbc = 0 \rangle$$

where $w(a) = w(b) = w(c) = 1$.

Now, we have the monoid algebra $A := k[\tilde{M}]$ over a field k:

$$A = \langle a, b, c \mid b^2 a = ab^2,\ b^2 c = aca,\ acc = 0,\ aba = 0,\ abc = 0,\ cba = 0,\ cbc = 0 \rangle$$

where $deg(a) = deg(b) = deg(c) = 1$ To show that A has intermediate growth, we first find a complete rewriting system for A. Let $\prec$ be the shortlex order on $\langle X \rangle$ based on the order $a \prec b \prec c$ i.e.,

$$w_1 \prec w_2 \text{ implies } |w_1| < |w_2| \text{ or } |w_1| = |w_2| \ \&\ w_1 \prec_{lex} w_2.$$

Then A has the rewriting system R consisting of the following relations

$$
\begin{aligned}
b^2 a &\rightarrow ab^2 \\
b^2 c &\rightarrow aca \\
acc &\rightarrow 0 \\
aba &\rightarrow 0 \\
abc &\rightarrow 0 \\
cba &\rightarrow 0 \\
cbc &\rightarrow 0
\end{aligned}
$$

It is easily seen that R is Noetherian. By applying the *Knuth-Bendix algorithm*, we obtain the following complete rewriting system R_∞ equivalent to R:

$$R_\infty = \{b^2a \to ab^2,\ b^2c \to aca,\ aba \to 0,\ abc \to 0,\ cba \to 0,\ cbc \to 0\} \cup \bigcup_{n=1}^{\infty} \{a^n ca^{n-1}c \to 0\}$$

A monomial (word) m is called *irreducible* with respect to the rewriting system R if all the rewriting rules act trivially on m. The set of all irreducible words with respect to R is denoted by $Irr(R)$. Since R_∞ is a complete rewriting system, $Irr(R_\infty)$ is the set of words which do not contain u as a subword for any $u \to v \in R_\infty$. By *Bergman's Diamond Lemma* [Ber78], $Irr(R_\infty)$, forms a basis for A. Words in $Irr(R_\infty)$ are of the following form

$$b^s a^{m_1} c a^{m_2} c \ldots a^{m_r} c a^l b^k$$

where $s \in \{0,1\}$, $l, k \in \mathbb{N} \cup \{0\}$ and $0 \le m_1 \le m_2 \le \cdots \le m_r$, $m_i \in \mathbb{N} \cup \{0\}$ for $i \in \{1, \ldots r\}$. So, the number of words in $Irr(R^\infty)$ of length n is equal to

$$\sum_{j=0}^{n} (2j+1) \cdot |\{(m_1, \ldots, m_r) \mid 0 \le m_1 \le \cdots \le m_r,\ m_1 + \cdots + m_r = n - j - r\}|$$

$$= \sum_{j=0}^{n} (2j+1) \cdot p(n-j)$$

where $p(n)$ is the number of partitions of n. Hence

$$\gamma_A(n) \sim p(n) \sim e^{\sqrt{n}}.$$

A is an example of finitely presented graded algebra with generators of degree 1 and intermediate growth function and its *Veronese subalgebra* $V_3(A)$ can be presented by finitely many quadratic relations (to be precise with 21 generators and 280 relations).

7   Appendix: Presentation of the Veronese subalgebra $V_4(U)$ of U

As we noted in the previous section, $U(L)$ is an associative algebra with generators x, y and the set of relations $R = \{x^3y - 3x^2yx + 3xyx^2 - yx^3 = 0,\ y^3x - 3y^2xy + 3yxy^2 - xy^3 = 0\}$. Since R is a set of two homogeneous polynomials, U is a graded algebra. Let $V_4(U)$ be the Veronese subalgebra of U. It was proven in Section 4 that $V_4(U)$ is a graded algebra generated by the set S of monomials of length 4 over $\{x, y\}$ and the set of relations $R^* = \{f_i = 0, v f_i w = 0\}$ where v, w are monomials such that $l(v) + l(w) = 4$ and, $f_1 = x^3y - 3x^2yx + 3xyx^2 - yx^3$, $f_2 = y^3x - 3y^2xy + 3yxy^2 - xy^3$. Basically, R^* is the set of homogeneous polynomials of degree 4 or 8 generated by $R = \{f_1 = 0, f_2 = 0\}$ in $k[x, y]$. Since there are 48 different pairs (v, w) of monomials, R^* consists of 2 homogeneous polynomials of degree 4:

$$(i)\ yx^3 = x^3y - 3x^2yx + 3xyx^2,\quad (ii)\ y^3x = xy^3 - 3yxy^2 + 3y^2xy$$

and 96 homogeneous polynomials of degree 8:

(1) $xyx^2x^4 = x^4yx^3 - 3x^3yx^4 + 3x^2yxx^4,$

(2) $x^3yx^4 = x^4x^2yx - 3x^4xyx^2 + 3x^4yx^3,$

(3) $x^2y^2yx^3 = x^3yy^2x^2 - 3x^2yxy^2x^2 + 3x^2y^2xyx^2,$

(4) $xyx^2x^3y = x^4yx^2y - 3x^3yx^3y + 3x^2yxx^3y,$

(5) $x^3yx^3y = x^4x^2y^2 - 3x^4xyxy + 3x^4yx^2y,$

(6) $x^2y^2yx^2y = x^3yy^2xy - 3x^2yxy^2xy + 3x^2y^2xyxy,$

(7) $xyx^2x^2yx = x^4yxyx - 3x^3yx^2yx + 3x^2yxx^2yx,$

(8) $x^2y^2x^4 = x^2yxx^2yx - 3x^2yxxyx^2 + 3x^2yxyx^3,$

(9) $x^2y^2yxyx = x^3yy^3x - 3x^2yxy^3x + 3x^2y^2xy^2x,$

(10) $xyx^2x^2y^2 = x^4yxy^2 - 3x^3yx^2y^2 + 3x^2yxx^2y^2,$

(11) $x^2y^2x^3y = x^2yxx^2y^2 - 3x^2yxxyxy + 3x^2yxyx^2y,$

(12) $x^2y^2yxy^2 = x^3yy^4 - 3x^2yxy^4 + 3x^2y^2xy^3,$

(13) $xyx^2xyx^2 = x^4y^2x^2 - 3x^3yxyx^2 + 3x^2yxxyx^2,$

(14) $xyxyx^4 = xyx^2x^2yx - 3xyx^2xyx^2 + 3xyx^2yx^3,$

(15) $xy^3yx^3 = xyxyy^2x^2 - 3xy^2xy^2x^2 + 3xy^3xyx^2,$

(16) $xyx^2xy^2x = x^4y^3x - 3x^3yxy^2x + 3x^2yxxy^2x,$

(17) $xy^3x^4 = xy^2xx^2yx - 3xy^2xxyx^2 + 3xy^2yx^3,$

(18) $xy^3yxyx = xyxyy^3x - 3xy^2xy^3x + 3xy^3xy^2x,$

(19) $xyx^2xyxy = x^4y^2xy - 3x^3yxyxy + 3x^2yxxyxy,$

(20) $xyxyx^3y = xyx^2x^2y^2 - 3xyx^2xyxy + 3xyx^2yx^2y,$

(21) $xy^3yx^2y = xyxyy^2xy - 3xy^2xy^2xy + 3xy^3xyxy,$

(22) $xyx^2xy^3 = x^4y^4 - 3x^3yxy^3 + 3x^2yxxy^3,$

(23) $xy^3x^3y = xy^2xx^2y^2 - 3xy^2xxyxy + 3xy^2xyx^2y,$

(24) $xy^3yxy^2 = xyxyy^4 - 3xy^2xy^4 + 3xy^3xy^3,$

(25) $y^2x^2x^4 = yx^3yx^3 - 3yx^2yx^4 + 3yxyxx^4,$

(26) $yx^2yx^4 = yx^3x^2yx - 3yx^3xyx^2 + 3yx^3yx^3,$

(27) $yxy^2yx^3 = yx^2yy^2x^2 - 3yxyxy^2x^2 + 3yxy^2xyx^2,$

(28) $x^2y^2x^2yx = yx^3yxyx - 3yx^2yx^2yx + 3yxyxyx^2xy,$

(29) $yxy^2x^4 = yxyxx^2yx - 3yxyxxyx^2 + 3yxyxyx^3,$

(30) $yxy^2yxyx = yx^2yy^3x - 3yxyxy^3x + 3yxy^2xy^2x,$

(31) $y^2x^2x^2y^2 = yx^3yxy^2 - 3yx^2yx^2y^2 + 3yxyxx^2y^2,$

(32) $yxy^2x^3y = yxyxx^2y^2 - 3yxyxxyxy + 3yxyxyx^2y,$

(33) $yxy^2yxy^2 = yx^2yy^4 - 3yxyxy^4 + 3yxy^2xy^3,$

(34) $y^2x^2x^3y = yx^3yx^2y - 3yx^2yx^3y + 3yxyxx^3y,$

(35) $yx^2yx^3y = yx^3x^2y^2 - 3yx^3xyxy + 3yx^3yx^2y,$

(36) $yxy^2yx^2y = yx^2yy^2xy - 3yxyxy^2xy + 3yxy^2xyxy,$

(37) $y^2x^2xyx^2 = yx^3y^2x^2 - 3yx^2yxyx^2 + 3yxyxxyx^2,$

(38) $y^2xyx^4 = y^2x^2x^2yx - 3y^2x^2xyx^2 + 3y^2x^2yx^3,$

(39) $y^4yx^3 = y^2xyy^2x^2 - 3y^3xy^2x^2 + 3y^4xyx^2,$

(40) $y^2x^2xyxy = yx^3y^2xy - 3yx^2yxyxy + 3yxyxxyxy,$

(41) $y^2xyx^3y = y^2x^2x^2y^2 - 3y^2x^2xyxy + 3y^2x^2yx^2y,$

(42) $y^4yx^2y = y^2xyy^2xy - 3y^3xy^2xy + 3y^4xyxy,$

(43) $y^2x^2xy^2x = yx^3y^3x - 3yx^2yxy^2x + 3yxyxxy^2x,$

(44) $y^4x^4 = y^3xx^2yx - 3y^3xxyx^2 + 3y^3xyx^3,$

(45) $y^4yxyx = y^2xyy^3x - 3y^3xy^3x + 3y^4xy^2x,$

(46) $y^2x^2xy^3 = yx^3y^4 - 3yx^2yxy^3 + 3yxyxxy^3,$

(47) $y^4x^3y = y^3xx^2y^2 - 3y^3xxyxy + 3y^3xyx^2y,$

(48) $y^4yxy^2 = y^2xyy^4 - 3y^3xy^4 + 3y^4xy^3,$

(49) $x^2yxx^4 = x^4xyx^2 - 3x^4yx^3 + 3x^3yx^4,$

(50) $xy^3x^4 = x^2y^2yx^3 - 3xyxyyx^3 + 3xy^2xyx^3,$

(51) $x^3yy^2x^2 = x^4y^3x - 3x^3yxy^2x + 3x^3yyxyx,$

(52) $x^2yxx^3y = x^4xyxy - 3x^4yx^2y + 3x^3yx^3y,$

(53) $xy^3x^3y = x^2y^2yx^2y - 3xyxyyx^2y + 3xy^2xyx^2y,$

(54) $x^3yy^2xy = x^4y^4 - 3x^3yxy^3 + 3x^3yyxy^2,$

(55) $x^2yxx^2yx = x^4xy^2x - 3x^4yxyx + 3x^3yx^2yx,$

(56) $xy^3x^2yx = x^2y^2yxyx - 3xyxyyxyx + 3xy^2xyxyx,$

(57) $x^2y^2y^2x^2 = x^2yxy^3x - 3x^2y^2xy^2x + 3x^2y^2yxyx,$

(58) $x^2yxx^2y^2 = x^4xy^3 - 3x^4yxy^2 + 3x^3yx^2y^2,$

(59) $xy^3x^2y^2 = x^2y^2yxy^2 - 3xyxyyxy^2 + 3xy^2xyxy^2,$

(60) $x^2y^2y^2xy = x^2yxy^4 - 3x^2y^2xy^3 + 3x^2y^2yxy^2,$

(61) $xy^2xx^4 = xyx^2xyx^2 - 3xyx^2yx^3 + 3xyxyx^4,$

(62) $xy^3xyx^2 = x^2y^2y^2x^2 - 3xyxyy^2x^2 + 3xy^2xy^2x^2,$

(63) $xyxyy^2x^2 = xyx^2y^3x - 3xyxyxy^2x + 3xyxyyxyx,$

(64) $xy^2xx^2yx = xyx^2xy^2x - 3xyx^2yxyx + 3xyxyx^2yx,$

(65) $xy^3xy^2x = x^2y^2y^3x - 3xyxyy^3x + 3xy^2xy^3x,$

(66) $xy^3y^2x^2 = xy^2xy^3x - 3xy^3xy^2x + 3xy^3yxyx,$

(67) $xy^2xx^3y = xyx^2xyxy - 3xyx^2yx^2y + 3xyxyx^3y,$

(68) $xy^3xyxy = x^2y^2y^2xy - 3xyxyy^2xy + 3xy^2xy^2xy,$

(69) $xyxyy^2xy = xyx^2y^4 - 3xyxyxy^3 + 3xyxyyxy^2,$

(70) $xy^2xx^2y^2 = xyx^2xy^3 - 3xyx^2yxy^2 + 3xyxyx^2y^2,$

(71) $xy^3xy^3 = x^2y^2y^4 - 3xyxyy^4 + 3xy^2xy^4,$

(72) $xy^3y^2xy = xy^2xy^4 - 3xy^3xy^3 + 3xy^3yxy^2,$

(73) $yxyxx^4 = yx^3xyx^2 - 3yx^3yx^3 + 3yx^2yx^4,$

(74) $y^4x^4 = yxy^2yx^3 - 3y^2xyyx^3 + 3y^3xyx^3,$

(75) $yx^2yy^2x^2 = yx^3y^3x - 3yx^2yxy^2x + 3yx^2yyxyx,$

(76) $yxyxx^2yx = yx^3xy^2x - 3yx^3yxyx + 3yx^2yx^2yx,$

(77) $y^4x^2yx = yxy^2yxyx - 3y^2xyyxyx + 3y^3xyxyx,$

(78) $yxy^2y^2x^2 = yxyxy^3x - 3yxy^2xy^2x + 3yxy^2yxyx,$

(79) $yxyxx^2y^2 = yx^3xy^3 - 3yx^3yxy^2 + 3yx^2yx^2y^2,$

(80) $y^4x^2y^2 = yxy^2yxy^2 - 3y^2xyyxy^2 + 3y^3xyxy^2,$

(81) $yxy^2y^2xy = yxyxy^4 - 3yxy^2xy^3 + 3yxy^2yxy^2,$

(82) $yxyxx^3y = yx^3xyxy - 3yx^3yx^2y + 3yx^2yx^3y,$

(83) $y^4x^3y = yxy^2yx^2y - 3y^2xyyx^2y + 3y^3xyx^2y,$

(84) $yx^2yy^2xy = yx^3y^4 - 3yx^2yxy^3 + 3yx^2yyxy^2,$

(85) $y^3xx^4 = y^2x^2xyx^2 - 3y^2x^2yx^3 - 3y^2xyx^4,$

(86) $y^4xyx^2 = yxy^2y^2x^2 - 3y^2xyy^2x^2 + 3y^3xy^2x^2,$

(87) $y^2xyy^2x^2 = y^2x^2y^3x - 3y^2xyxy^2x + 3y^2xyyxyx,$

(88) $y^3xx^3y = y^2x^2xyxy - 3y^2x^2yx^2y + 3y^2xyx^3y,$

(89) $y^4xyxy = yxy^2y^2xy - 3y^2xyy^2xy + 3y^3xy^2xy,$

(90) $y^2xyy^2xy = y^2x^2y^4 - 3y^2xyxy^3 + 3y^2xyyxy^2,$

(91) $y^3xx^2yx = y^2x^2yx^2x - 3y^2x^2yxyx + 3y^2xyx^2yx,$

(92) $y^4xy^2x = yxy^2y^3x - 3y^2xyy^3x + 3y^3xy^3x,$

(93) $y^4y^2x^2 = y^3xy^3x - 3y^4xy^2x + 3y^4yxyx,$

(94) $y^3xx^2y^2 = y^2x^2xy^3 - 3y^2x^2yxy^2 + 3y^2xyx^2y^2,$

(95) $y^4xy^3 = yxy^2y^4 - 3y^2xyy^4 - 3y^2xyy^4 + 3y^3xy^4,$

(96) $y^4y^2xy = y^3xy^4 - 3y^4xy^3 + 3y^4yxy^2.$

We can rename the generators as follows:

$$y^4 = Y_1, \ y^3x = Y_2, \ y^2xy = Y_3, \ y^2x^2 = Y_4, \ yxy^2 = Y_5, \ yxyx = Y_6, yx^2y = Y_7, \ yx^3 = Y_8,$$

$$xy^3 = X_1, \ xy^2x = X_2, \ xyxy = X_3, \ xyx^2 = X_4, \ x^2y^2 = X_5, \ x^2yx = X_6, \ x^3y = X_7, \ x^4 = X_8.$$

So the relations will be

$$(i) \ Y_8 = X_7 - 3X_6 + 3X_4, \quad (ii) \ Y_2 = X_1 - 3Y_5 + 3Y_3$$

(1) $X_4X_8 = X_8Y_8 - 3X_7X_8 + 3X_6X_8,$

(2) $X_7X_8 = X_8X_6 - 3X_8X_4 + 3X_8Y_8,$

(3) $X_5Y_8 = X_7Y_4 - 3X_6Y_4 + 3X_5X_4,$

(4) $X_4X_7 = X_8Y_7 - 3X_7X_7 + 3X_6X_7,$

(5) $X_7X_7 = X_8X_5 - 3X_8X_3 + 3X_8Y_7,$

(6) $X_5Y_7 = X_7Y_3 - 3X_6Y_3 + 3X_5X_3,$

(7) $X_4X_6 = X_8Y_6 - 3X_7X_6 + 3X_6X_6,$

(8) $X_5X_8 = X_6X_6 - 3X_6X_4 + 3X_6Y_8,$

(9) $X_5Y_6 = X_7Y_2 - 3X_6Y_2 + 3X_5X_2,$

(10) $X_4X_5 = X_8Y_5 - 3X_7X_5 + 3X_6X_5,$

(11) $X_5X_7 = X_6X_5 - 3X_6X_3 + 3X_6Y_7,$

(12) $X_5Y_5 = X_7Y_1 - 3X_6Y_1 + 3X_5X_1,$

(13) $X_4X_4 = X_8Y_4 - 3X_7X_4 + 3X_6X_4,$

(14) $X_3X_8 = X_4X_6 - 3X_4X_4 + 3X_4Y_8,$

(15) $X_1Y_8 = X_3Y_4 - 3X_2Y_4 + 3X_1X_4,$

(16) $X_4X_2 = X_8Y_2 - 3X_7X_2 + 3X_6X_2,$

(17) $X_1X_8 = X_2X_6 - 3X_2X_4 + 3X_2Y_8,$

(18) $X_1Y_6 = X_3Y_2 - 3X_2Y_2 + 3X_1X_2,$

(19) $X_4X_3 = X_8Y_3 - 3X_7X_3 + 3X_6X_3,$

(20) $X_3X_7 = X_4X_5 - 3X_4X_3 + 3X_4Y_7,$

(21) $X_1Y_7 = X_3Y_3 - 3X_2Y_3 + 3X_1X_3,$

(22) $X_4X_1 = X_8Y_1 - 3X_7X_1 + 3X_6X_1,$

(23) $X_1X_7 = X_2X_5 - 3X_2X_3 + 3X_2Y_7,$

(24) $X_1Y_5 = X_3Y_1 - 3X_2Y_1 + 3X_1X_1,$

(25) $Y_4X_8 = Y_8Y_8 - 3Y_7X_8 + 3Y_6X_8,$

(26) $Y_7X_8 = Y_8X_6 - 3Y_8X_4 + 3Y_8Y_8,$

(27) $Y_5Y_8 = Y_7Y_4 - 3Y_6Y_4 + 3Y_5X_4,$

(28) $Y_4X_6 = Y_8Y_6 - 3Y_7X_6 + 3Y_6X_6,$

(29) $Y_5X_8 = Y_6X_6 - 3Y_6X_4 + 3Y_6Y_8,$

(30) $Y_5Y_6 = Y_7Y_2 - 3Y_6Y_2 + 3Y_5X_2,$

(31) $Y_4X_5 = Y_8Y_5 - 3Y_7X_5 + 3Y_6X_5,$

(32) $Y_5X_7 = Y_6X_5 - 3Y_6X_3 + 3Y_6Y_7,$

(33) $Y_5Y_5 = Y_7Y_1 - 3Y_6Y_1 + 3Y_5X_1,$

(34) $Y_4X_7 = Y_8Y_7 - 3Y_7X_7 + 3Y_6X_7,$

(35) $Y_7X_7 = Y_8X_5 - 3Y_8X_3 + 3Y_8Y_7,$

(36) $Y_5Y_7 = Y_7Y_3 - 3Y_6Y_3 + 3Y_5X_3,$

(37) $Y_4X_4 = Y_8Y_4 - 3Y_7X_4 + 3Y_6X_4,$

(38) $Y_3X_8 = Y_4X_6 - 3Y_4X_4 + 3Y_4Y_8,$

(39) $Y_1Y_8 = Y_3Y_4 - 3Y_2Y_4 + 3Y_1X_4,$

(40) $Y_4X_3 = Y_8Y_3 - 3Y_7X_3 + 3Y_6X_3,$

(41) $Y_3X_7 = Y_4X_5 - 3Y_4X_3 + 3Y_4Y_7,$

(42) $Y_1Y_7 = Y_3Y_3 - 3Y_2Y_3 + 3Y_1X_3,$

(43) $Y_4X_2 = Y_8Y_2 - 3Y_7X_2 + 3Y_6X_2,$

(44) $Y_1X_8 = Y_2X_6 - 3Y_2X_4 + 3Y_2Y_8,$

(45) $Y_1Y_6 = Y_3Y_2 - 3Y_2Y_2 + 3Y_1X_2,$

(46) $Y_4X_1 = Y_8Y_1 - 3Y_7X_1 + 3Y_6X_1,$

(47) $Y_1X_7 = Y_2X_5 - 3Y_2X_3 + 3Y_2Y_7,$

(48) $Y_1Y_5 = Y_3Y_1 - 3Y_2Y_1 + 3Y_1X_1,$

(49) $X_6X_8 = X_8X_4 - 3X_8Y_8 + 3X_7X_8,$

(50) $X_1X_8 = X_5Y_8 - 3X_3Y_8 + 3X_2Y_8,$

(51) $X_7Y_4 = X_8Y_2 - 3X_7X_2 + 3X_7Y_6,$

(52) $X_6X_7 = X_8X_3 - 3X_8Y_7 + 3X_7X_7,$

(53) $X_1X_7 = X_5Y_7 - 3X_3Y_7 + 3X_2Y_7,$

(54) $X_7Y_3 = X_8Y_1 - 3X_7X_1 + 3X_7Y_5,$

(55) $X_6X_6 = X_8X_2 - 3X_8Y_6 + 3X_7X_6,$

(56) $X_1X_6 = X_5Y_6 - 3X_3Y_6 + 3X_2Y_6,$

(57) $X_5Y_4 = X_6Y_2 - 3X_5X_2 + 3X_5Y_6,$

(58) $, X_6X_5 = X_8X_1 - 3X_8Y_5 + 3X_7X_5$

(59) $X_1X_5 = X_5Y_5 - 3X_3Y_5 + 3X_2Y_5,$

(60) $X_5Y_3 = X_6Y_1 - 3X_5X_1 + 3X_5Y_5,$

(61) $X_2X_8 = X_4X_4 - 3X_4Y_8 + 3X_3X_8,$

(62) $X_1X_4 = X_5Y_4 - 3X_3Y_4 + 3X_2Y_4,$

(63) $X_3Y_4 = X_4Y_2 - 3X_3X_2 + 3X_3Y_6,$

(64) $X_2X_6 = X_4X_2 - 3X_4Y_6 + 3X_3X_6,$

(65) $X_1X_2 = X_5Y_2 - 3X_3Y_2 + 3X_2Y_2,$

(66) $X_1Y_4 = X_2Y_2 - 3X_1X_2 + 3X_1Y_6,$

(67) $X_2X_7 = X_4X_3 - 3X_4Y_7 + 3X_3X_7,$

(68) $X_1X_3 = X_5Y_3 - 3X_3Y_3 + 3X_2Y_2,$

(69) $X_3Y_3 = X_4Y_1 - 3X_3X_1 + 3X_3Y_5,$

(70) $X_2X_5 = X_4X_1 - 3X_4Y_5 + 3X_3X_5,$

(71) $X_1X_1 = X_5Y_1 - 3X_3Y_1 + 3X_2Y_1,$

(72) $X_1Y_3 = X_2Y_1 - 3X_1X_1 + 3X_1Y_5,$

(73) $Y_6X_8 = Y_8X_4 - 3Y_8Y_8 + 3Y_7X_8,$

(74) $Y_1X_8 = Y_5Y_8 - 3Y_3Y_8 + 3Y_2Y_8,$

(75) $Y_7Y_4 = Y_8Y_2 - 3Y_7X_2 + 3Y_7Y_6,$

(76) $Y_6X_6 = Y_8X_2 - 3Y_8Y_6 + 3Y_7X_6,$

(77) $Y_1X_6 = Y_5Y_6 - 3Y_3Y_6 + 3Y_2Y_6,$

(78) $Y_5Y_4 = Y_6Y_2 - 3Y_5X_2 + 3Y_5Y_6,$

(79) $Y_6X_5 = Y_8X_1 - 3Y_8Y_5 + 3Y_7X_5$

(80) $Y_1X_5 = Y_5Y_5 - 3Y_3Y_5 + 3Y_2Y_5,$

(81) $Y_5Y_3 = Y_6Y_1 - 3Y_5X_1 + 3Y_5Y_5,$

(82) $Y_6X_7 = Y_8X_3 - 3Y_8Y_7 + 3Y_7X_7,$

(83) $Y_1X_7 = Y_5Y_7 - 3Y_3Y_7 + 3Y_2Y_7,$

(84) $Y_7Y_3 = Y_8Y_1 - 3Y_7X_1 + 3Y_7Y_5,$

(85) $Y_2X_8 = Y_4X_4 - 3Y_4Y_8 + 3Y_3X_8,$

(86) $Y_1X_4 = Y_5Y_4 - 3Y_3Y_4 + 3Y_2Y_4,$

(87) $Y_3Y_4 = Y_4Y_2 - 3Y_3X_2 + 3Y_3Y_6,$

(88) $Y_2X_7 = Y_4X_3 - 3Y_4Y_7 + 3Y_3X_7,$

(89) $Y_1X_3 = Y_5Y_3 - 3Y_3Y_3 + 3Y_2Y_3,$

(90) $Y_3Y_3 = Y_4Y_1 - 3Y_3X_1 + 3Y_3Y_5,$

(91) $Y_2X_6 = Y_4X_2 - 3Y_4Y_6 + 3Y_3X_6,$

(92) $Y_1X_2 = X_5Y_2 - 3Y_3Y_2 + 3Y_2Y_2,$

(93) $Y_1Y_4 = Y_2Y_2 - 3Y_1X_2 + 3Y_1Y_6,$

(94) $Y_2X_5 = Y_4X_1 - 3Y_4Y_5 + 3Y_3X_5,$

(95) $Y_1X_1 = Y_5Y_1 - 3Y_3Y_1 + 3Y_2Y_1,$

(96) $Y_1Y_3 = Y_2Y_1 - 3Y_1X_1 + 3Y_1Y_5.$

We see that $V_4(U)$ is a quadratic algebra with generators $X_1, \ldots, X_8, Y_1, \ldots Y_8$ and relations $(i), (ii)$, $(1) - (96)$. This may not be the simplest presentation of $V_4(U)$. Observe that the generators Y_8 and Y_2 are linear combinations of other generators by (i) and (ii), so they can be removed from the generating set.

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